Construction of Hypergeometric Solutions to the q-Painlevé Equations

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Abstract

Hypergeometric solutions to the q-Painlevé equations are constructed by direct linearization of disrcrete Riccati equations. The decoupling factors are explicitly determined so that the linear systems give rise to q-hypergeometric equations.

1 Introduction

This article is a continuation of our previous work [1] on the hypergeometric solutions to the q-Painlevé equations in the following degeneration diagram of affine Weyl group symmetries [2, 3] :

$$E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)} \to D_5^{(1)} \to A_4^{(1)} \to (A_2+A_1)^{(1)} \to (A_1+\underset{|\alpha|^2=14}{A_1})^{(1)} \tag{1}$$

The list of q-Painlevé equations we are going to investigate is given in section 3 below. We remark that these q-Painlevé equations were discovered through various approaches to discrete Painlevé equations, including singularity confinement analysis, compatibility conditions of linear difference equations, affine Weyl group symmetries and τ-functions on the lattices. Also, in Sakai's framework [2], each of these q-Painlevé equations is constructed in a unified manner as the birational action of a translation of the corresponding affine Weyl group on a certain family of rational surfaces.

In [4] we have introduced the formulation of discrete Painlevé equations based on the geometry of plane curves on $\mathbb{P}^2$. On that basis we were able in the first part of [1] to find suitable coordinates for linearization of the q-Painlevé equations into three-term relations of hypergeometric functions. As a result we obtained the following degeneration diagram of basic hypergeometric functions corresponding to (1):

$$\begin{matrix}\text{balanced}\\ {}_{10}W_9\end{matrix} \to {}_8W_7 \to \begin{matrix}\text{balanced}\\ {}_3\varphi_2\end{matrix} \to {}_2\varphi_1 \to {}_1\varphi_1 \to \begin{matrix}{}_1\varphi_1\left(\begin{matrix}a\\0\end{matrix};q,z\right)\\ {}_1\varphi_1\left(\begin{matrix}0\\b\end{matrix};q,z\right)\end{matrix} \to {}_1\varphi_1\left(\begin{matrix}0\\-q\end{matrix};q,z\right)$$

This shows the usefulness of geometric consideration in the study of particular solutions of discrete Painleé equations. In order to determine explicit solutions to those equations written in the literature, further steps of precise computations are required since the variables to be solved are fixed in advance. We have shown only the results for them in the second part of [1]. The purpose of this article is to present these explicit solutions including subtle gauge factors and to show the calculation in detail based on the direct linearization of discrete Riccati equations.

This article is organized as follows: In section 2, we explain the procedure to construct hypergeometric solutions through the linearization of discrete Riccati equations. We demonstrate this procedure by taking the case of $E_7^{(1)}$ as an example. In section 3, we give the list of q-Painlevé equations and their hypergeometric solutions, with the data that are necessary for constructing solutions. Among the cases in the diagram (1) we have excluded the case of $D_5^{(1)}$, since a complete identification of hypergeometric solutions has already been given in [5].

2 Construction of Hypergeometric Solutions

2.1 Discrete Riccati Equation and Its Linearization

The q-Painlevé equations admit particular solutions characterized by discrete Riccati equations for special values of parameters. Reduction to discrete Riccati equation has been already done for all the q-Painlevé equations[3, 6, 7, 8]. We also note that such special situations have clear geometrical meaning, as was discussed in [1]. The basic idea for constructing hypergeometric solution is as follows: we linearize the discrete Riccati equations to yield second order linear q-difference equations. We then identify them with the three-term relation of an appropriate basic hypergeometric series.

Let us explain this procedure in detail. Suppose we have a discrete Riccati equation of the form

$$\overline{z} = \frac{Az + B}{Cz + D}, \tag{2}$$

where $z = z(t)$ and $\overline{z} = z(qt)$. We also use the notation $\underline{z} = z(t/q)$, and so forth. Moreover, the coefficients A, B, C and D are functions of t. First let us put an ansatz

$$z = \frac{F}{G}. \tag{3}$$

Then the discrete Riccati equation is linearized to

$$\frac{\overline{F}}{H} = AF + BG, \quad \frac{\overline{G}}{H} = CF + DG, \tag{4}$$

where H is an arbitrary decoupling factor. Eliminating G from eq.(4) we have for F the three-term relation

$$\overline{F} + c_1 F + c_2 \underline{F} = 0, \quad c_1 = -\frac{H}{\underline{B}}(A\underline{B} + B\underline{D}), \quad c_2 = \frac{B}{\underline{B}} H\underline{H}(\underline{AD} - \underline{BC}). \tag{5}$$

The three-term relation for a basic hypergeometric series often takes the form

$$V_1(\overline{\Phi} - \Phi) + V_2\Phi + V_3(\underline{\Phi} - \Phi) = 0, \tag{6}$$

where the coefficients V_1, V_2 and V_3 are factorized into binomials involving the independent variable and parameters. Comparing eqs.(5) with (6), we have

$$\frac{V_2}{V_1} = 1 + c_1 + c_2, \quad \frac{V_3}{V_2} = c_2. \tag{7}$$

We look for the decoupling factor H so that these quantities factorize. We then identify the three-term relation with that for appropriate hypergeometric function. This is done by trial and error with the aid of computer algebra, but it is not practically difficult since we already know the hypergeometric function and its three-term relation to appear for each q-Painlevé equation.

Step 1. *Find the decoupling factor H such that*

$$\frac{V_2}{V_1} = 1 + c_1 + c_2, \quad \frac{V_3}{V_2} = c_2, \tag{8}$$

factorize. Then identify the three-term relation

$$V_1(\overline{F} - F) + V_2 F + V_3(\underline{F} - F) = 0, \tag{9}$$

with that for an appropriate hypergeometric function.

Similarly, we have for G the three-term relation

$$\overline{G} + \tilde{d}_1 G + \tilde{d}_2 \underline{G} = 0, \quad \tilde{d}_1 = -\frac{H}{\underline{C}}(D\underline{C} + C\underline{A}), \quad \tilde{d}_2 = \frac{C}{\underline{C}} H\underline{H}(\underline{AD} - \underline{BC}). \tag{10}$$

However, usually $1+\tilde{d}_1+\tilde{d}_2$ does not factorize for H obtained above. Replacing G with κG ($\overline{\kappa}=k\kappa$), we have

$$z=\frac{1}{\kappa}\frac{F}{G}, \tag{11}$$

$$\overline{G}+d_1G+d_2\underline{G}=0,\quad d_1=-\frac{H}{k\underline{C}}(D\underline{C}+C\underline{A}),\quad d_2=\frac{C}{\underline{C}}\frac{H\underline{H}}{k\underline{k}}(\underline{AD}-\underline{BC}). \tag{12}$$

We then look for k so that $1+d_1+d_2$ factorizes, and identify eq.(12) with the three-term relation of the same hypergeometric function as F with different parameters. Putting $H/k=K$, this is equivalent to the follwing procedure:

Step 2. *In the three-term relation*

$$\overline{G}+d_1G+d_2\underline{G}=0,\quad d_1=-\frac{K}{\underline{C}}(D\underline{C}+C\underline{A}),\quad d_2=\frac{C}{\underline{C}}K\underline{K}(\underline{AD}-\underline{BC}), \tag{13}$$

find decoupling factor K so that

$$\frac{U_2}{U_1}=1+d_1+d_2,\quad \frac{U_3}{U_2}=d_2, \tag{14}$$

factorize. Then identify the three-term relation with that for an appropriate hypergeometric function. Now we have

$$z=\frac{1}{\kappa}\frac{\theta_1\Phi}{\theta_2\Psi},\quad \frac{H}{K}=k,\quad \frac{\overline{\kappa}}{\kappa}=k, \tag{15}$$

where Φ and Ψ are some hypergeometric functions, and θ_i ($i=1,2$) are constants(gauge factors).

Finally we determine the gauge factors θ_1 and θ_2:

Step 3. *Compare the linear relations,*

$$\frac{\theta_1\overline{\Phi}}{H}=A\theta_1\Phi+\kappa B\theta_2\Psi,\quad \frac{\overline{\kappa}\theta_2\overline{\Psi}}{H}=C\theta_1\Phi+\kappa D\theta_2\Psi, \tag{16}$$

with contiguity relations of the hypergeometric functions to determine θ_1 and θ_2.

2.2 An Example: Case of $E_7^{(1)}$

In this section we demonstrate the construction of the hypergeometric solution to the q-Painlevé equation of type $E_7^{(1)}$ as an example, following the procedure in the previous section.

Before proceeding, let us first summarize the definition and terminology of the basic hypergeometric series [9]. The basic hypergeometric series ${}_r\varphi_s$ is given by

$$\begin{gathered}{}_r\varphi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,z\right)=\sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_r;q)_n}{(b_1;q)_n\cdots(b_s;q)_n(q;q)_n}\left[(-1)^nq^{\binom{n}{2}}\right]^{1+s-r}z^n,\\ (a;q)_n=(1-a)(1-qa)\cdots(1-q^{n-1}a).\end{gathered} \tag{17}$$

The basic hypergeometric series ${}_{r+1}\varphi_r$ is called *balanced*[1] if the condition

$$qa_1a_2\cdots a_{r+1}=b_1b_2\cdots b_r,\quad z=q, \tag{18}$$

is satisfied, and is called *very-well-poised* if the condition

$$qa_1=a_2b_1=\cdots=a_{r+1}b_r,\quad a_2=qa_1^{\frac{1}{2}},\quad a_3=-qa_1^{\frac{1}{2}}, \tag{19}$$

[1] The "balanced ${}_3\varphi_2$" in the diagram (1) is due to the convention that was used in [10].

is satisfied. A very-well-poised hypergeometric series ${}_{r+1}\varphi_r$ is denoted as ${}_{r+1}W_r$:

$$ {}_{r+1}W_r(a_1; a_4, \dots, a_{r+1}; q, z) = {}_r\varphi_s\left(\begin{array}{c} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4 \dots, a_{r+1} \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4, \dots, qa_1/a_{r+1} \end{array}; q, z\right). \tag{20} $$

Now the q-Painlevé equation of type $E_7^{(1)}$ is given by [2, 3]

$$ \begin{aligned} \frac{(f\overline{g} - \overline{t}t)(fg - t^2)}{(f\overline{g} - 1)(fg - 1)} &= \frac{(f - b_1t)(f - b_2t)(f - b_3t)(f - b_4t)}{(f - b_5)(f - b_6)(f - b_7)(f - b_8)}, \\ \frac{(fg - t^2)(\underline{f}g - t\underline{t})}{(fg - 1)(\underline{f}g - 1)} &= \frac{\left(g - \frac{t}{b_1}\right)\left(g - \frac{t}{b_2}\right)\left(g - \frac{t}{b_3}\right)\left(g - \frac{t}{b_4}\right)}{\left(g - \frac{1}{b_5}\right)\left(g - \frac{1}{b_6}\right)\left(g - \frac{1}{b_7}\right)\left(g - \frac{1}{b_8}\right)}, \end{aligned} \tag{21} $$

where t is the independent variable and b_i $(i = 1, \dots, 8)$ are parameters satisfying

$$ \overline{t} = qt, \quad b_1b_2b_3b_4 = q, \quad b_5b_6b_7b_8 = 1. \tag{22} $$

Proposition 2.1 *[3, 6] In case of $b_1b_3 = b_5b_7$, eq.(21) admits a specialization to the discrete Riccati equation,*

$$ \overline{g} = \frac{(t\overline{t} - 1)f + t\left\{-(b_6 + b_8)\overline{t} + (b_2 + b_4)\right\}}{\{-(b_6 + b_8) + (b_2 + b_4)t\}\, f + b_6b_8(1 - t\overline{t})}, \tag{23} $$

$$ f = \frac{(t^2 - 1)b_5b_7g + t\,\{(b_1 + b_3) - (b_5 + b_7)t\}}{\{t(b_1 + b_3) - (b_5 + b_7)\}\, g + (1 - t^2)}. \tag{24} $$

As was pointed out in [1], in the cases of $E_{6,7,8}^{(1)}$ the variables f and g are not expressed by ratio of hypergeometric functions. We choose the variable as[2]

$$ z = \frac{g - t/b_1}{g - 1/b_5}. \tag{25} $$

Then the discrete Riccati equation (23) and (24) is rewritten as

$$ \overline{z} = \frac{Az + B}{Cz + D}, $$

with

$$ \begin{aligned} A &= b_1b_5(-b_3 + b_5t) \\ &\quad \times\Big[b_4b_6b_8q^2t^3 + (b_1b_4b_5 - b_4^2b_5 - b_1b_4b_6 - b_1b_4b_8 - b_5b_6b_8q)qt^2 \\ &\qquad + (b_1b_4^2 + b_4b_5b_6q + b_4b_5b_8q + b_1b_6b_8q - b_4b_6b_8q)t - b_1b_4b_5\Big], \end{aligned} \tag{26} $$

$$ B = -(b_1 - b_4)b_5^2(b_1b_4 - qb_6b_8)t(b_5 - b_3t)(-1 + qt^2), \tag{27} $$

$$ C = -b_1^2b_4(b_5 - b_6)(b_5 - b_8)(b_3 - b_5t)(-1 + qt^2), \tag{28} $$

$$ \begin{aligned} D &= b_1b_5(b_5 - b_3t) \\ &\quad \times\Big[-b_1b_4b_5qt^3 + (b_1b_4^2 + b_4b_5b_6q + b_4b_5b_8q + b_1b_6b_8q - b_4b_6b_8q)t^2 \\ &\qquad + (b_1b_4b_5 - b_4^2b_5 - b_1b_4b_6 - b_1b_4b_8 - b_5b_6b_8q)t + b_4b_6b_8\Big]. \end{aligned} \tag{29} $$

[2] This variable z should be understood as a ratio of τ functions.

Step 1. We choose the decoupling factor H as

$$H = \frac{1}{qb_1b_5(b_5t - b_3)(b_1t - b_5)(b_4t - b_6)(b_4t - b_8)}. \tag{30}$$

Then we have for F the three-term relation

$$\begin{aligned} &V_1(\overline{F} - F) + V_2F + V_3(\underline{F} - F) = 0,\\ &\frac{V_2}{V_1} = \frac{b_5(b_3 - b_4)(b_1b_4 - qb_6b_8)(-1 + t)(1 + t)(-1 + qt^2)}{q(b_5 - b_1t)(-b_6 + b_4t)(-b_8 + b_4t)(-b_3 + b_5t)},\\ &\frac{V_3}{V_1} = \frac{(b_5 - b_3t)(-b_1 + b_5t)(-b_4 + b_6t)(-b_4 + b_8t)(-1 + qt^2)}{(b_5 - b_1t)(-b_6 + b_4t)(-b_8 + b_4t)(-b_3 + b_5t)(-q + t^2)}. \end{aligned} \tag{31}$$

The three-term relation for the very-well-poised basic hypergeometric series

$$\Phi = {}_8W_7\left(a_0; a_1, a_2, a_3, a_4, a_5; q, \frac{q^2a_0^2}{a_1a_2a_3a_4a_5}\right), \tag{32}$$

is given by [11]

$$\begin{aligned} &U_1(\overline{\Phi} - \Phi) + U_2\Phi + U_3(\underline{\Phi} - \Phi) = 0, \qquad \overline{\Phi} = \Phi\,|_{a_2\to qa_2, a_3\to a_3/q}, \quad \underline{\Phi} = \Phi\,|_{a_2\to a_2/q, a_3\to qa_3},\\ &U_1 = \frac{(1 - a_2)\left(1 - \frac{a_0}{a_2}\right)\left(1 - \frac{qa_0}{a_2}\right)\left(1 - \frac{qa_0}{a_1a_3}\right)\left(1 - \frac{qa_0}{a_3a_4}\right)\left(1 - \frac{qa_0}{a_3a_5}\right)}{a_3\left(1 - \frac{a_2}{a_3}\right)\left(1 - \frac{qa_2}{a_3}\right)}, \quad U_3 = U_1|_{a_2\leftrightarrow a_3},\\ &U_2 = \frac{qa_0^2}{a_1a_2a_3a_4a_5}\left(1 - \frac{qa_0}{a_2a_3}\right)(1 - a_1)(1 - a_4)(1 - a_5). \end{aligned} \tag{33}$$

Comparing eqs.(31) with (33), we identify F with ${}_8W_7$ as

$$F \propto {}_8W_7\left(\frac{b_1b_8}{b_3b_5}; \frac{qb_8}{b_5}, \frac{b_1t}{b_5}, \frac{b_1}{b_5t}, \frac{b_2}{b_3}, \frac{b_4}{b_3}; q, \frac{b_5}{b_6}\right). \tag{34}$$

Step 2. We choose the decoupling factor K as

$$K = \frac{1}{b_1b_5(qb_5t - b_3)(b_1t - b_5)(b_4t - b_6)(b_4t - b_8)}. \tag{35}$$

Then we have for G the three-term relation

$$\begin{aligned} &X_1(\overline{G} - G) + X_2F + X_3(\underline{G} - G) = 0,\\ &\frac{X_2}{X_1} = \frac{b_5(b_3 - b_4)(b_1b_4 - b_6b_8)(-1 + t)(1 + t)(-1 + qt^2)}{(b_5 - b_1t)(-b_6 + b_4t)(-b_8 + b_4t)(-b_3 + qb_5t)},\\ &\frac{X_3}{X_1} = \frac{(qb_5 - b_3t)(-b_1 + b_5t)(-b_4 + b_6t)(-b_4 + b_8t)(-1 + qt^2)}{(b_5 - b_1t)(-b_6 + b_4t)(-b_8 + b_4t)(-b_3 + qb_5t)(-q + t^2)}. \end{aligned} \tag{36}$$

Comparing eqs.(36) with (33), we have

$$G \propto {}_8W_7\left(\frac{b_1b_8}{b_3b_5}; \frac{b_8}{b_5}, \frac{b_1t}{b_5}, \frac{b_1}{b_5t}, \frac{b_2}{b_3}, \frac{b_4}{b_3}; q, \frac{qb_5}{b_6}\right). \tag{37}$$

Moreover, from $k = H/K = (1 - b_3/qb_5t)/(1 - b_3/b_5t)$, we have $\kappa = 1 - b_3/b_5t$. Therefore we obtain

$$z \propto \frac{1}{1 - \frac{b_3}{b_5t}}\ \frac{{}_8W_7\left(a_0; qa_1, a_2, a_3, a_4, a_5; q, \frac{qa_0^2}{a_1a_2a_3a_4a_5}\right)}{{}_8W_7\left(a_0, a_1, a_2, a_3, a_4, a_5; q, \frac{q^2a_0^2}{a_1a_2a_3a_4a_5}\right)}, \tag{38}$$

with

$$a_0=\frac{b_1b_8}{b_3b_5},\quad a_1=\frac{b_8}{b_5},\quad a_2=\frac{b_1t}{b_5},\quad a_3=\frac{b_1}{b_5t},\quad a_4=\frac{b_2}{b_3},\quad a_5=\frac{b_4}{b_3},\quad \frac{q^2a_0^2}{a_1a_2a_3a_4a_5}=\frac{qb_5}{b_6}. \tag{39}$$

Step 3. Let us put

$$F=\theta(qa_1,a_2,a_3)\Phi(qa_1,a_2,a_3),\quad G=\theta(a_1,a_2,a_3)\Phi(a_1,a_2,a_3), \tag{40}$$

where $\theta(a_1,a_2,a_3)$ is a gauge factor to be determined. Here, we have omitted the dependence of a_0,a_4 and a_5, since they are not relevant to the calculation. Then linear equations (16) yield

$$\frac{1}{\kappa HB}\frac{\theta(qa_1,qa_2,a_3/q)}{\theta(a_1,a_2,a_3)}\,\Phi(qa_1,qa_2,a_3/q)=\frac{A}{\kappa B}\frac{\theta(qa_1,a_2,a_3)}{\theta(a_1,a_2,a_3)}\,\Phi(qa_1,a_2,a_3)+\Phi(a_1,a_2,a_3), \tag{41}$$

and

$$\frac{\overline{\kappa}}{\kappa HD}\frac{\theta(a_1,qa_2,a_3/q)}{\theta(a_1,a_2,a_3)}\,\Phi(a_1,qa_2,a_3/q)=\frac{C}{\kappa D}\frac{\theta(qa_1,a_2,a_3)}{\theta(a_1,a_2,a_3)}\,\Phi(qa_1,a_2,a_3)+\Phi(a_1,a_2,a_3), \tag{42}$$

respectively. Now, we have the contiguity relations for $\Phi={}_8W_7$ [11]

$$\begin{aligned}&\frac{a_1\left(1-\frac{a_0q}{a_1a_3}\right)\left(1-\frac{a_0q}{a_1a_4}\right)\left(1-\frac{a_0q}{a_1a_5}\right)}{1-\frac{a_0q}{a_1}}\,\Phi(a_1/q,a_2,a_3)-\Big(a_1\leftrightarrow a_2\Big)\\&=(a_1-a_2)\left(1-\frac{a_0^2q^2}{a_1a_2a_3a_4a_5}\right)\Phi(a_1,a_2,a_3),\end{aligned} \tag{43}$$

$$\begin{aligned}&(a_2-1)\left(1-\frac{a_0}{a_2}\right)\Phi(a_1/q,qa_2,a_3)+\left(1-\frac{a_1}{q}\right)\left(1-\frac{a_0q}{a_1}\right)\Phi(a_1,a_2,a_3)\\&=\left(a_2-\frac{a_1}{q}\right)\left(1-\frac{a_0q}{a_1a_2}\right)\Phi(a_1/q,a_2,a_3).\end{aligned} \tag{44}$$

We denote eqs.(43) and (44) as CR1$[a_1,a_2,a_3]$ and CR2$[a_1,a_2,a_3]$, respectively. Moreover, note that the relations CR1$[a_1,a_2,a_3]$ and CR2$[a_1,a_2,a_3]$ hold for any permutation of a_1,a_2 and a_3, since these parameters are on equal footing in $\Phi={}_8W_7$.

Now we eliminate $\Phi(a_1,a_2,a_3/q)$ from CR1$[a_1,a_3,a_2]$ and CR2$[a_3,a_2,a_1]$. Shifting a_1 to qa_1, we have a linear relation among $\Phi(a_1,qa_2,a_3/q)$, $\Phi(a_1,a_2,a_3)$ and $\Phi(a_1/q,a_2,a_3)$, which should coincide with eq. (41). Similarly, eliminating $\Phi(qa_1,a_2,a_3/q)$ from CR1$[qa_1,a_3,a_2]$ and CR2$[a_3,a_1,a_2]$, we have a linear relation among $\Phi(qa_1,a_2,a_3)$, $\Phi(a_1,a_2,a_3)$ and $\Phi(a_1,a_2,a_3/q)$. Elimination further $\Phi(a_1,a_2,a_3/q)$ from this relation and CR2$[a_3,a_2,a_1]$ yields a linear relation among $\Phi(a_1,qa_2,a_3/q)$, $\Phi(qa_1,a_2,a_3)$ and $\Phi(a_1,a_2,a_3)$, which should coincide with eq.(42). From these calculations, we find that $\theta(a_1,a_2,a_3)$ should satisfy

$$\begin{aligned}&\frac{\theta(a_1,qa_2,a_3/q)}{\theta(a_1,a_2,a_3)}=\frac{\left(1-\frac{a_2}{a_0}\right)\left(1-\frac{a_3a_5}{a_0q}\right)}{\left(1-\frac{a_2a_5}{a_0}\right)\left(1-\frac{a_3}{a_0q}\right)}=\frac{\left(1-\frac{b_3t}{b_8}\right)\left(1-\frac{b_4}{b_8qt}\right)}{\left(1-\frac{b_4t}{b_8}\right)\left(1-\frac{b_3}{b_8qt}\right)},\\&\frac{\theta(qa_1,a_2,a_3)}{\theta(a_1,a_2,a_3)}=1-\frac{a_1}{a_0}=1-\frac{b_3}{b_1},\\&\frac{\theta(qa_1,qa_2,a_3/q)}{\theta(a_1,a_2,a_3)}=\frac{\theta(a_1,qa_2,a_3/q)}{\theta(a_1,a_2,a_3)}\times\frac{\theta(qa_1,a_2,a_3)}{\theta(a_1,a_2,a_3)},\end{aligned} \tag{45}$$

which yield

$$\theta(a_1,a_2,a_3)=\frac{\left(\frac{a_2}{a_0q},\frac{a_3a_5}{a_0q},\frac{a_1}{a_0q}\right)_\infty}{\left(\frac{a_2a_5}{aq},\frac{a_3}{a_0q}\right)_\infty}=\frac{\left(\frac{b_3t}{qb_8},\frac{b_4}{qb_8t},\frac{b_3}{qb_1}\right)_\infty}{\left(\frac{b_4t}{qb_8},\frac{b_3}{qb_8t}\right)_\infty}. \tag{46}$$

Therefore we arrive at the final result

$$
\begin{aligned}
z &= \frac{1}{1-\dfrac{b_3}{b_5 t}}\,\frac{\theta(qa_1,a_2,a_3)}{\theta(a_1,a_2,a_3)}\,\frac{\Phi(qa_1,a_2,a_3)}{\Phi(a_1,a_2,a_3)} \\
&= \frac{1-\dfrac{b_3}{b_1}}{1-\dfrac{b_3}{b_5 t}}\,\frac{{}_8W_7\left(a_0;qa_1,a_2,a_3,a_4,a_5;q,\dfrac{qa_0^2}{a_1a_2a_3a_4a_5}\right)}{{}_8W_7\left(a_0,a_1,a_2,a_3,a_4,a_5;q,\dfrac{q^2a_0^2}{a_1a_2a_3a_4a_5}\right)}.
\end{aligned}
\tag{47}
$$

3 Hypergeometric Solutions

Hypergeometric solutions to other q-Painlevé equations can be constructed by the same procedure as that was demonstrated in the previous section. Instead of describing full procedure, we give a list of equations, solutions and the other data that are necessary for construction of solutions. We note that the case of $D_5^{(1)}$ is omitted as mentioned in the introduction.

3.1 Case of $E_8^{(1)}$

3.1.1 Equation and Solution

(1) q-Painlevé equation [3, 6, 12]

$$
\begin{aligned}
&\frac{(\overline{g}\overline{s}t-f)(gst-f)-(\overline{s}^2t^2-1)(s^2t^2-1)}{\left(\dfrac{\overline{g}}{\overline{s}t}-f\right)\left(\dfrac{g}{st}-f\right)-\left(1-\dfrac{1}{\overline{s}^2t^2}\right)\left(1-\dfrac{1}{s^2t^2}\right)} = \frac{P(f,t,m_1,\ldots,m_7)}{P(f,t^{-1},m_7,\ldots,m_1)},\\
&\frac{(\underline{f}s\underline{t}-g)(fst-g)-(s^2\underline{t}^2-1)(s^2t^2-1)}{\left(\dfrac{\underline{f}}{s\underline{t}}-g\right)\left(\dfrac{f}{st}-g\right)-\left(1-\dfrac{1}{s^2\underline{t}^2}\right)\left(1-\dfrac{1}{s^2t^2}\right)} = \frac{P(g,s,m_7,\ldots,m_1)}{P(g,s^{-1},m_1,\ldots,m_7)},
\end{aligned}
\tag{48}
$$

where

$$
\begin{aligned}
P(f,t,m_1,\ldots,m_7) = {}& f^4-m_1tf^3+(m_2t^2-3-t^8)f^2\\
&+(m_7t^7-m_3t^3+2m_1t)f+(t^8-m_6t^6+m_4t^4-m_2t^2+1),
\end{aligned}
\tag{49}
$$

and m_k ($k=1,2,\ldots 7$) are the elementary symmetric functions of k-th degree in b_i ($i=1,2,\ldots,8$) with

$$b_1b_2\cdots b_8 = 1. \tag{50}$$

Moreover,

$$\overline{t} = qt,\ \ t = q^{\frac{1}{2}}s. \tag{51}$$

(2) Constraint on parameters [6]

$$qb_1b_3b_5b_7 = 1, \quad b_2b_4b_6b_8 = q. \tag{52}$$

(3) Hypergeometric solution

$$
z = \frac{g-\left(\dfrac{s}{b_1}+\dfrac{b_1}{s}\right)}{g-\left(\dfrac{s}{b_8}+\dfrac{b_8}{s}\right)} = \lambda\,\frac{\Phi(q^4a_0;a_1,q^2a_2,\ldots,q^2a_7)}{\Phi(a_0;a_1\ldots,a_7)},
\tag{53}
$$

where Φ is defined in terms of the balanced ${}_{10}W_9$ series by

$$\begin{aligned}\Phi(a_0;a_1,\ldots,a_7) &= {}_{10}W_9(a_0;a_1,\ldots,a_7;q^2,q^2)\\ &+\frac{\left(q^2a_0,\dfrac{a_7}{a_0};q^2\right)_\infty}{\left(\dfrac{a_0}{a_7},\dfrac{q^2a_7^2}{a_0};q^2\right)_\infty}\prod_{k=1}^{6}\frac{\left(a_k,\dfrac{q^2a_7}{a_k};q^2\right)_\infty}{\left(\dfrac{q^2a_0}{a_k},\dfrac{a_ka_7}{a_0};q^2\right)_\infty}\,{}_{10}W_9\left(\frac{a_7^2}{a_0};\frac{a_1a_7}{a_0},\ldots,\frac{a_6a_7}{a_0},a_7;q^2,q^2\right).\end{aligned} \tag{54}$$

Here, a_i ($i=0,1,\ldots,7$) and λ are given by

$$\begin{aligned}&a_0=\frac{1}{qb_1b_2b_8^2},\quad a_1=\frac{q^2}{b_2b_8t^2},\quad a_2=\frac{s^2}{b_2b_8},\\ &a_i=\frac{b_i}{b_8}\ (i=3,5,7),\quad a_i=\frac{b_i}{b_1}\ (i=4,6),\end{aligned} \tag{55}$$

and

$$\lambda=\frac{b_1b_4b_6}{b_8s^2}\,\frac{\left(1-\dfrac{b_4b_6}{b_1b_8}\right)\left(1-q^2\dfrac{b_4b_6}{b_1b_8}\right)(1-b_3b_5t^2)(1-b_3b_7t^2)(1-b_5b_7t^2)\displaystyle\prod_{i=2,4,6}\left(1-\frac{b_i}{b_1}\right)}{\left(1-\dfrac{s^2}{b_1b_8}\right)\left(1-\dfrac{q^2s^2}{b_1b_8}\right)\left(1-\dfrac{b_4}{b_8}\right)\left(1-\dfrac{b_6}{b_8}\right)\left(1-\dfrac{q}{b_1b_8s^2}\right)\displaystyle\prod_{i=3,5,7}\left(1-\frac{b_4b_6}{b_1b_i}\right)}, \tag{56}$$

respectively.

3.1.2 Data

(1) Riccati equation

$$\begin{vmatrix}1&f&g&fg\\1&f_1&g_1&f_1g_1\\1&f_3&g_3&f_3g_3\\1&f_5&g_5&f_5g_5\end{vmatrix}=0,\quad \begin{vmatrix}1&f&\overline{g}&f\overline{g}\\1&f_8&\overline{g}_8&f_8\overline{g}_8\\1&f_6&\overline{g}_6&f_6\overline{g}_6\\1&f_4&\overline{g}_4&f_4\overline{g}_4\end{vmatrix}=0, \tag{57}$$

where

$$f_i=b_it+\frac{1}{b_it},\quad g_i=\frac{s}{b_i}+\frac{b_i}{s}. \tag{58}$$

The Riccati equation for $z=\dfrac{g-g_1}{g-g_8}$ is given by

$$\overline{z}=\frac{Az+B}{Cz+D}, \tag{59}$$

$$\begin{aligned}&B=-f_{35}g_{13}g_{15}d'_{1468},\\ &C=f_{46}\overline{g}_{48}\overline{g}_{68}d_{1358},\\ &D=-f_{35}f_{46}f_{18}g_{13}g_{15}\overline{g}_{48}\overline{g}_{68},\\ &\Delta=AD-BC=f_{13}f_{35}f_{15}f_{46}f_{48}f_{68}g_{13}g_{15}g_{35}g_{18}\overline{g}_{46}\overline{g}_{48}\overline{g}_{68}\overline{g}_{18},\end{aligned} \tag{60}$$

where $f_{ij}=f_i-f_j$ and

$$d_{1358}=\begin{vmatrix}1&f_1&g_1&f_1g_1\\1&f_3&g_3&f_3g_3\\1&f_5&g_5&f_5g_5\\1&f_8&g_8&f_8g_8\end{vmatrix},\quad d'_{1468}=\begin{vmatrix}1&f_1&\overline{g}_1&f_1\overline{g}_1\\1&f_4&\overline{g}_4&f_4\overline{g}_4\\1&f_6&\overline{g}_6&f_6\overline{g}_6\\1&f_8&\overline{g}_8&f_8\overline{g}_8\end{vmatrix}, \tag{61}$$

respectively.

(2) Three-term and contiguity relations for hypergeometric function [13]

(a) Three-term relation

$$U_1(\overline{\Phi}-\Phi)+U_2\Phi+U_3(\underline{\Phi}-\Phi)=0, \tag{62}$$

where

$$U_1=\frac{a_1(1-a_2)\left(1-\dfrac{a_0}{a_2}\right)\left(1-\dfrac{q^2a_0}{a_2}\right)}{(1-\dfrac{q^2a_2}{a_1})\displaystyle\prod_{j=3}^{7}\left(1-\dfrac{q^2a_0}{a_1a_j}\right)},$$

$$U_2=-(a_1-a_2)\left(1-\frac{q^2a_0}{a_1a_2}\right)\prod_{j=3}^{7}(1-a_j),\quad U_3=U_1|_{a_1\leftrightarrow a_2}, \tag{63}$$

$$\overline{\Phi}=\Phi(a_0;a_1/q^2,q^2a_2,a_3,\ldots,a_7),\quad \underline{\Phi}=\Phi(a_0;q^2a_1,a_2/q^2,a_3,\ldots,a_7), \tag{64}$$

and Φ is defined by eq.(54).

(b) Contiguity relations

$$\begin{aligned}&\Phi(a_0;a_1/q^2,q^2a_2,a_3,\ldots,a_7)-\Phi(a_0;a_1,a_2,a_3,\ldots,a_7)\\&=V_1\,\Phi(q^4a_0^2;a_1,q^2a_2,\ldots,q^2a_7),\end{aligned} \tag{65}$$

$$\begin{aligned}&V_2\,\Phi(q^4a_0^2;a_1,q^2a_2,q^2a_3,\ldots,q^2a_7)-V_3\,\Phi(q^4a_0^2;q^2a_1,a_2,q^2a_3,\ldots,q^2a_7)\\&=V_4\,\Phi(a_0;a_1,a_2,a_3,\ldots,a_7),\end{aligned} \tag{66}$$

where

$$V_1=\frac{\dfrac{q^2a_0}{a_2}\left(1-\dfrac{q^2a_2}{a_1}\right)\left(1-\dfrac{a_1a_2}{q^2a_0}\right)(1-q^2a_0)(1-q^4a_0)\displaystyle\prod_{j=3}^{7}(1-a_j)}{\left(1-\dfrac{q^2a_0}{a_1}\right)\left(1-\dfrac{q^4a_0}{a_1}\right)\left(1-\dfrac{a_0}{a_2}\right)\left(1-\dfrac{q^2a_0}{a_2}\right)\displaystyle\prod_{j=3}^{7}\left(1-\dfrac{q^2a_0}{a_j}\right)},$$

$$V_2=\frac{a_1^2(1-a_2)\displaystyle\prod_{j=3}^{7}\left(1-\dfrac{q^2a_0}{a_1a_j}\right)}{\left(1-\dfrac{q^2a_0}{a_1}\right)\left(1-\dfrac{q^4a_0}{a_1}\right)},\quad V_3=V_2|_{a_1\leftrightarrow a_2}, \tag{67}$$

$$V_4=\frac{a_1\left(1-\dfrac{a_2}{a_1}\right)\displaystyle\prod_{j=3}^{7}\left(1-\dfrac{q^2a_0}{a_j}\right)}{(1-q^2a_0)(1-q^4a_0)}.$$

(3) Decoupling factors

$$H=\frac{D}{\Delta}=-\frac{f_{18}}{f_{13}f_{15}f_{48}f_{68}g_{35}g_{18}\overline{g}_{46}\overline{g}_{18}},\quad K=\frac{1}{D}=-\frac{1}{f_{35}f_{46}f_{18}g_{13}g_{15}\overline{g}_{48}\overline{g}_{68}}, \tag{68}$$

so that

$$k=\frac{H}{K}=\frac{D^2}{\Delta}=\frac{f_{35}f_{46}f_{18}^2g_{13}g_{15}\overline{g}_{48}\overline{g}_{68}}{f_{13}f_{15}f_{48}f_{68}g_{35}\overline{g}_{46}\overline{g}_{18}}. \tag{69}$$

(4) Identification

$$z=\frac{1}{\kappa}\,\frac{F}{G},\quad F\propto\Phi(q^4a_0;a_1,q^2a_2,\ldots,q^2a_7),\quad G\propto\Phi(a_0;a_1\ldots,a_7),\quad \overline{\kappa}=k\kappa, \tag{70}$$

where a_i $(i=0,\ldots,7)$ are given by eq.(55).

(5) Gauge factors
Putting

$$\begin{aligned}
F &= \theta(q^4a_0; a_1, q^2a_2, \ldots, q^2a_7)\Phi(q^4a_0; a_1, q^2a_2, \ldots, q^2a_7),\\
G &= \theta(a_0; a_1 \ldots, a_7)\Phi(a_0; a_1 \ldots, a_7),
\end{aligned} \tag{71}$$

we have:

$$\frac{\theta(a_0; a_1/q^2, a_2q^2, \ldots, a_7)}{\theta(a_0; a_1 \ldots, a_7)} = 1, \quad \frac{1}{\kappa}\frac{\theta(q^4a_0; a_1, q^2a_2, \ldots, q^2a_7)}{\theta(a_0; a_1 \ldots, a_7)} = \lambda. \tag{72}$$

3.2 Case of $E_6^{(1)}$

3.2.1 Equation and Solution

(1) q-Painlevé equation[3, 14, 15]

$$\begin{aligned}
(f\overline{g}-1)(fg-1) &= t\bar{t}\,\frac{(f-b_1t)(f-b_2t)(f-b_3t)(f-b_4t)}{(f-b_5t)\left(f-\dfrac{t}{b_5}\right)},\\
(fg-1)(\underline{f}g-1) &= t^2\,\frac{\left(g-\dfrac{1}{b_1}\right)\left(g-\dfrac{1}{b_2}\right)\left(g-\dfrac{1}{b_3}\right)\left(g-\dfrac{1}{b_4}\right)}{(g-b_6t)\left(g-\dfrac{t}{b_6}\right)},\\
\bar{t} &= qt, \quad b_1b_2b_3b_4 = 1.
\end{aligned} \tag{73}$$

(2) Constraint on parameters [3, 15]

$$b_1b_2 = b_5b_6. \tag{74}$$

(3) Hypergeometric solution

$$z = \frac{g-\dfrac{1}{b_1}}{g-tb_6} = \frac{1-\dfrac{b_3}{b_1}}{1-\dfrac{b_1b_2b_3t}{b_5}}\,\frac{\Phi(qa,b,c,d,e)}{\Phi(a,b,qc,d,e)}, \tag{75}$$

where Φ is the balanced ${}_3\varphi_2$ series defined by

$$\Phi(a,b,c,d,e) = {}_3\varphi_2\left(\begin{matrix} a,b,c\\ d,e\end{matrix}\;;q,\frac{de}{abc}\right), \tag{76}$$

with

$$a = \frac{b_3b_5}{t}, \quad b = \frac{b_3}{b_2}, \quad c = b_1^2b_2b_3, \quad d = q\frac{b_3b_5^2}{b_2}, \quad e = qb_1b_2b_3^2. \tag{77}$$

3.2.2 Data

(1) Riccati equation [3, 15]

$$\overline{g} = \frac{1+\dfrac{b_5\bar{t}}{b_1b_2}(f-b_1-b_3)}{f-\bar{t}b_5}, \quad f = \frac{1+b_6t(b_3b_4g-b_3-b_4)}{g-tb_6}. \tag{78}$$

The Riccati equation for

$$z = \frac{g-1/b_1}{g-tb_6}, \tag{79}$$

is given by,

$$\overline{z} = \frac{Az+B}{Cz+D},$$

$$\begin{aligned}
A &= b_1b_5(b_3b_5-t)(b_5-b_1b_2b_3t)(-b_2+b_5qt),\\
B &= -b_5^2t(b_1-b_3)(-1+b_1^2b_2b_3)(-b_2+b_5qt),\\
C &= qtb_1(b_1b_2-b_5)(b_1b_2+b_5)(b_3b_5-t)(-b_5+b_1b_2b_3t),\\
D &= -b_5\Big[b_1b_2b_3b_5^2+(-b_1^3b_2^2b_3b_5-b_1^3b_2^2b_3b_5q-b_2b_3b_5^3q)t\\
&\quad +(b_1^3b_2^2-b_1^2b_2^2b_3+b_1^4b_2^3b_3^2-b_1b_5^2+b_3b_5^2+b_1^3b_2b_3b_5^2+b_1^2b_2^2b_3b_5^2-b_1^2b_2b_3^2b_5^2+b_1^2b_2^2b_3b_5^2q)qt^2\\
&\quad -b_1^4b_2^3b_3b_5q^2t^3\Big].
\end{aligned} \tag{80}$$

(2) Three-term and contiguity relations for hypergeometric function [10]

(a) Three-term relation

$$V_1\left(\overline{\Phi}-\Phi\right)+V_2\Phi+V_3\left(\Phi-\underline{\Phi}\right)=0, \tag{81}$$

where

$$\begin{aligned}
&V_1=\left(1-\frac{q}{z}\right)(1-a)\,,\quad V_2=(1-b)(1-c),\quad V_3=\frac{a}{z}\left(1-\frac{d}{a}\right)\left(1-\frac{e}{a}\right),\\
&\Phi=\Phi(a,b,c,d,e)={}_3\varphi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q;z\right),\quad z=\frac{de}{abc},\quad \overline{\Phi}=\Phi|_{a\to qa}\,,\quad \underline{\Phi}=\Phi|_{a\to a/q}\,.
\end{aligned} \tag{82}$$

(b) Contiguity relations

$$(a-c)\Phi(a,b,c,d,e)+(1-a)\Phi(qa,b,c,d,e)-(1-c)\Phi(a,b,qc,d,e)=0, \tag{83}$$

$$\begin{aligned}
&(a-c)(de-abc)\Phi(a,b,c,d,e)+bc(d-a)(e-a)\Phi(a/q,b,c,d,e)\\
&-ab(d-c)(e-c)\Phi(a,b,c/q,d,e)=0.
\end{aligned} \tag{84}$$

(3) Decoupling factors

$$\begin{aligned}
H &= \frac{1}{b_1b_3b_5(-b_5+b_1^2b_2t)(b_5-b_1b_2b_3t)(b_2-qb_5t)},\\
K &= \frac{1}{b_1b_3b_5(-b_5+b_1^2b_2t)(b_5-qb_1b_2b_3t)(b_2-qb_5t)},\\
k &= \frac{H}{K}=\frac{b_5-qb_1b_2b_3t}{b_5-b_1b_2b_3t},\quad \kappa=1-\frac{b_1b_2b_3t}{b_5}.
\end{aligned} \tag{85}$$

(4) Identification

$$z=\frac{1}{\kappa}\,\frac{F}{G},\quad F\propto\Phi(qa,b,c,d,e),\quad G\propto\Phi(a,b,qc,d,e), \tag{86}$$

where $a,\ldots,e$ are given by eq.(77).

(5) Gauge factors
Putting

$$F=\theta(qa,b,c,d,e)\Phi(qa,b,c,d,e),\quad G=\theta(a,b,qc,d,e)\Phi(a,b,qc,d,e), \tag{87}$$

we have:

$$\frac{\theta(a,b,c,d,e)}{\theta(a,b,qc,d,e)}=\frac{\theta(qa,b,c,d,e)}{\theta(a,b,qc,d,e)}=1-\frac{b_3}{b_1},\quad \frac{\theta(a/q,b,qc,d,e)}{\theta(a,b,qc,d,e)}=1. \tag{88}$$

3.3 Case of $A_4^{(1)}$

3.3.1 Equation and Solution

(1) q-Painlevé equation [3]

$$\overline{g}g = \frac{\left(f+\frac{a_1}{t}\right)\left(f+\frac{1}{a_1 t}\right)}{1+a_3 f},$$
$$f\underline{f} = \frac{\left(g+\frac{a_2}{s}\right)\left(g+\frac{1}{a_2 s}\right)}{1+g/a_3},\tag{89}$$
$$\overline{t} = qt, \quad t = q^{\frac{1}{2}} s.$$

(2) Constraint on parameters [3]

$$a_1 a_2 a_3^2 = q^{-\frac{1}{2}}.\tag{90}$$

(3) Hypergeometric solution

$$g = -\frac{1}{a_1 a_3^2 t}\,\frac{\Phi(\alpha_1,\alpha_2,z)}{\Phi(\alpha_1,q\alpha_2,z)}, \quad f = \frac{1}{a_3}\left(1-\frac{1}{a_2^2}\right)\,\frac{\Phi(q\alpha_1,q\alpha_2,z)}{\Phi(\alpha_1,q\alpha_2,z)},\tag{91}$$

where Φ is the ${}_2\varphi_1$ series defined by

$$\Phi(\alpha_1,\alpha_2,z) = {}_2\phi_1\left(\begin{matrix}\alpha_1,\alpha_2\\ 0\end{matrix};q,z\right),\tag{92}$$

with

$$\alpha_1 = \frac{1}{a_2^2}, \quad \alpha_2 = a_1^2, \quad z = \frac{t}{a_1 a_3}.\tag{93}$$

Note that the solution is also expressible in terms of ${}_1\phi_1$ series by using the formula [16],

$${}_2\phi_1\left(\begin{matrix}a,b\\ 0\end{matrix};q,z\right) = \frac{(bz;q)_\infty}{(z;q)_\infty}\,{}_1\phi_1\left(\begin{matrix}b\\ bz\end{matrix};q,az\right).\tag{94}$$

3.3.2 Data

(1) Riccati equation [3]

$$\overline{g} = \frac{a_3^2\, g + \frac{1-a_1^2}{a_1 t}}{-a_3 g + \left(\frac{1}{a_3^2} - \frac{1}{a_1 a_3 t}\right)}, \quad g = -\frac{f+\frac{1}{a_1 t}}{a_3^2}.\tag{95}$$

(2) Three-term and contiguity relations for hypergeometric function

(a) Three-term relation

$$\frac{\alpha_1\alpha_2}{q}z\,\left(\overline{\Phi}-\Phi\right) + \frac{z}{q}(1-\alpha_1)(1-\alpha_2)\,\Phi + \left(\frac{z}{q}-1\right)\,(\underline{\Phi}-\Phi) = 0,\tag{96}$$

where

$$\Phi(\alpha_1,\alpha_2,z) = {}_2\phi_1\left(\begin{matrix}\alpha_1,\alpha_2\\ 0\end{matrix};q,z\right), \quad \overline{\Phi} = \Phi|_{z\to qz}, \quad \underline{\Phi} = \Phi|_{z\to z/q}.\tag{97}$$

(b) Contiguity relations

$$\Phi(\alpha_1,\alpha_2,z)-\alpha_2\Phi(\alpha_1,\alpha_2,qz)=(1-\alpha_2)\,\Phi(\alpha_1,q\alpha_2,z), \tag{98}$$

$$\Phi(\alpha_1,\alpha_2/q,z)-\Phi(\alpha_1,\alpha_2,z)=-\frac{\alpha_2 z}{q}(1-\alpha_1)\,\Phi(q\alpha_1,\alpha_2,z), \tag{99}$$

$$\alpha_1\alpha_2 z\,\Phi(\alpha_1,q\alpha_2,qz)=\Phi-(1-\alpha_2 z)\,\Phi(\alpha_1,q\alpha_2,z). \tag{100}$$

(3) Decoupling factors

$$H=\frac{1}{a_1^2a_3^2},\quad K=\frac{1}{qa_1^2a_3^2},\quad k=\frac{H}{K}=q,\quad \kappa=t. \tag{101}$$

(4) Identification

$$f=\frac{1}{\kappa}\,\frac{F}{G},\quad F\propto\Phi(\alpha_1,\alpha_2,z),\quad G\propto\Phi(\alpha_1,q\alpha_2,z), \tag{102}$$

with parameters given in eq.(93).

(5) Gauge factors

Putting

$$F=\theta(\alpha_1,\alpha_2,z)\Phi(\alpha_1,\alpha_2,z),\quad G=\theta(\alpha_1,q\alpha_2,z)\Phi(\alpha_1,q\alpha_2,z), \tag{103}$$

we have:

$$\frac{\theta(\alpha_1,\alpha_2,qz)}{\theta(\alpha_1,\alpha_2,z)}=1,\quad \frac{\theta(\alpha_1,q\alpha_2,z)}{\theta(\alpha_1,\alpha_2,z)}=-a_1a_3^2. \tag{104}$$

3.4 Case of $(A_2+A_1)^{(1)}$

3.4.1 Equation and Solution

(1) q-Painlevé equation [2, 3, 7, 8, 17]

$$\overline{g}gf=b_0\,\frac{1+a_0tf}{a_0t+f},\quad gf\underline{f}=b_0\,\frac{\dfrac{a_1}{t}+g}{1+\dfrac{a_1}{t}g},\quad \overline{t}=qt. \tag{105}$$

Eq. (105) admits two different specializations for hypergeometric solutions: (a) specialization of b_i (parameter of A_1), (b) specialization of a_i (parameter of A_2). See also [7, 8] for details.

(2) Constraint on parameters

(a)

$$b_0=q. \tag{106}$$

(b)

$$a_0a_1=q. \tag{107}$$

(3) Hypergeometric solution

(a)

$$g=-\frac{a_1}{t}\left(1-\frac{q^2}{a_0^2a_1^2}\right)\frac{\Phi(b,z)}{\Phi(q^2b,z)},\quad f=\frac{q^2t}{a_0a_1^2}\,\frac{1}{1-\dfrac{q^2}{a_0^2a_1^2}}\,\frac{\Phi(q^2b,q^2z)}{\Phi(b,z)}, \tag{108}$$

where

$$\Phi=\Phi(b,z)={}_1\varphi_1\left(\begin{matrix}0\\ b\end{matrix};q^2,z\right), \tag{109}$$

with

$$b=q^2/a_0^2a_1^2,\quad z=q^2t^2/a_1^2. \tag{110}$$

(b)

$$g = \frac{b_0}{a_0 t}\,\frac{\Psi(a,z)}{\Psi(a,q^2z)}, \quad f = -a_0 t\,\frac{\Psi(a,q^2z)}{\Psi(a,z)}, \tag{111}$$

where

$$\Psi = \Psi(a,z) = {}_1\varphi_1\left(\begin{matrix} a \\ 0 \end{matrix}\ ;q^2,z\right), \tag{112}$$

with

$$a = a_0^2t^2, \quad z = q/b_0. \tag{113}$$

3.4.2 Data

(1) Riccati equation

(a)

$$\overline{f} = \frac{\left(a_0^2a_1^2/q - a_0^2qt^2\right) f - a_0qt}{a_0tf+1}, \quad g = -a_0a_1\,\frac{1}{a_0t+f}. \tag{114}$$

(b)

$$\overline{g} = -\frac{g-a_0b_0t}{a_0tg-b_0}, \quad fg = -b_0. \tag{115}$$

(2) Three-term and contiguity relations for hypergeometric function

(a) (i) Three-term relation

$$\frac{b}{z}\left(\Phi(b,q^2z)-\Phi(b,z)\right)+\Phi(b,z)+\frac{q^2}{z}\left(\Phi(b,z/q^2)-\Phi\right)=0, \tag{116}$$

where

$$\Phi(b,z) = {}_1\varphi_1\left(\begin{matrix} 0 \\ b \end{matrix}\ ;q^2,z\right). \tag{117}$$

(ii) Contiguity relations

$$\Phi(b,z)-\frac{b}{q^2}\Phi(b,q^2z)=\left(1-\frac{b}{q^2}\right)\Phi(b/q^2,z), \tag{118}$$

$$\Phi(b,z)-\Phi(b,z/q^2)=\frac{z/q^2}{1-b}\,\Phi(q^2b,z). \tag{119}$$

(b) (i) Three-term relation

$$(1-a)\frac{z}{q^2}\left(\Psi(q^2a,z)-\Psi(a,z)\right)-\frac{az}{q^2}\Psi(a,z)+\left(\Psi(a/q^2,z)-\Psi(a,z)\right)=0, \tag{120}$$

where

$$\Psi(a,z) = {}_1\varphi_1\left(\begin{matrix} a \\ 0 \end{matrix}\ ;q^2,z\right). \tag{121}$$

(ii) Contiguity relations

$$\Psi(a,z)-a\Psi(a,q^2z)=(1-a)\Psi(q^2a,z), \tag{122}$$

$$\Psi(a,z)-\Psi(q,z/q^2)=(1-a)\frac{z}{q^2}\,\Psi(q^2a,z). \tag{123}$$

(3) Decoupling factors

(a)

$$H=\frac{1}{q},\quad K=1,\quad k=\frac{H}{K}=\frac{1}{q},\quad \kappa=\frac{a_1}{qt}. \tag{124}$$

(b)

$$H=-\frac{1}{1-a_0^2t^2},\quad K=\frac{1}{q}\,\frac{1}{1-a_0^2t^2},\quad k=\frac{H}{K}=q,\quad \kappa=a_0t. \tag{125}$$

(4) Identification

(a)

$$f=\frac{1}{\kappa}\,\frac{F}{G},\quad F\propto\Phi(q^2b,q^2z),\quad G\propto\Phi(b,z), \tag{126}$$

with parameters given in eq.(110).

(b)

$$g=\frac{1}{\kappa}\,\frac{F}{G},\quad F\propto\Psi(a,z),\quad G\propto\Psi(a,q^2z), \tag{127}$$

with parameters given in eq.(113).

(5) Gauge factors

(a) Putting

$$F=\theta(q^2b,q^2z)\Phi(q^2b,q^2z),\quad G=\theta(b,z)\Phi(b,z), \tag{128}$$

we have:

$$\frac{\theta(b,q^2z)}{\theta(b,z)}=1,\quad \frac{\theta(q^2b,z)}{\theta(b,z)}=\frac{\theta(q^2b,q^2z)}{\theta(b,z)}=\frac{q}{a_0a_1\left(1-\dfrac{q^2}{a_0^2a_1^2}\right)}. \tag{129}$$

(b) Putting

$$F=\theta(a,z)\Psi(a,z),\quad G=\theta(a,q^2z)\Psi(a,q^2z), \tag{130}$$

we have:

$$\frac{\theta(q^2a,z)}{\theta(a,z)}=1,\quad \frac{\theta(a,q^2z)}{\theta(a,z)}=\frac{\theta(q^2a,q^2z)}{\theta(a,z)}=\frac{1}{b_0}. \tag{131}$$

3.5 Case of $(A_1+A_1')^{(1)}$

3.5.1 Equation and Solution

(1) q-Painlevé equation [2, 3, 18]

$$(\overline{f}f-1)(f\underline{f}-1)=\frac{at^2f}{f+t},\quad \bar{t}=qt. \tag{132}$$

(2) Constraint on parameters

$$a=q. \tag{133}$$

(3) Hypergeometric solution

$$f=\frac{\Phi(qt)}{\Phi(t)},\quad \Phi={}_1\varphi_1\left(\begin{matrix}0\\-q\end{matrix};q,-qt\right). \tag{134}$$

3.5.2 Data

(1) Riccati equation

$$\overline{f} = \frac{1}{f} - qt. \tag{135}$$

(2) Three-term relation

$$\Phi(qt) + t\Phi(t) = \Phi(t/q). \tag{136}$$

(3) Identification

$$f = \frac{F}{G}, \quad F = \Phi(qt), \quad G = \Phi(t). \tag{137}$$

We note that there is no need to introduce decoupling and gauge factors.

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